測度0の鎖回帰集合をもつ写像の通用性について (一般位相幾何学及び幾何学的トポロジーの最近の話題とその応用)

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測度 0 の鎖回帰集合をもつ写像の通用性について

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本稿の目的は, 著者の論文 [15] の要約 (résumé) と若干の補足をすることであり, 証明などは原論文を参照ください。

1. 序

Chain recurrent points have been introduce by C. Conley [7]. They play an important role in the theory of attractors and in several other aspects of topological dynamics of a continuous map f on a compact metric space X. The key theorem here is Conley's Decomposition Theorem which says that the space X decomposes into the chain recurrent set $\mathrm{CR}(f)$ (see §2 for definition) and the rest, where the action is *gradient-like* (see [7] for definition). Note that the chain recurrent set contains all nonwandering points in that including the "genuine" recurrent points x (i.e., such that x belongs to the closure of its forward orbit), minimal subsets and periodic orbits.

Another motivation for studying chain recurrent sets in this particular context (of n-dimensional locally $(n-1)$-connected spaces) is provided by two other results: The first one is Pugh's Closing Lemma, which allows to replace chain recurrent points by periodic ones (by slightly perturbing the map):

Theorem ([13] for manifolds). *Let (X, d) be an n-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connectedness assumption), and $f : X \to X$ be a map. If $x \in \mathrm{CR}(f)$, then for every $\varepsilon > 0$, there exists a map $g : X \to X$ such that the uniform distance $d(f, g) < \varepsilon$ and x is a periodic point of g.*

Sketch of proof. We give here an outline in the case when X is n-dimensional locally $(n-1)$-connected, $n \in \mathbb{N}$. Let $x \in \mathrm{CR}(f)$, and any $\varepsilon > 0$ is given. We may assume $x \notin \mathrm{Per}(f)$.

Since X is locally $(n-1)$-connected, we have a ξ such that $0 < \xi < \varepsilon/2$ and

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(1) for every map $\varphi : A \to X$ from a closed set A of a compact metric space Z with $\dim Z \leq n$ and $\operatorname{diam}[\operatorname{Im}\varphi] < \xi$, there exists an extension $\tilde{\varphi} : Z \to X$ of φ satisfying $\operatorname{diam}[\operatorname{Im}\tilde{\varphi}] < \varepsilon/2$.

Using uniform continuity of f, we also take a $\delta > 0$ such that

(2) if $A \subseteq X$ with $\operatorname{diam}[A] < \delta$, then $\operatorname{diam}[f(A)] < \xi/2$.

Then take a $\xi/2$-chain $\{x_0 = x, x_1, \ldots, x_k = x\}$ of least possible length k; hence, $k \geq 1$ and $x_i \neq x_j$ for $0 \leq i < j \leq k-1$. We have an open neighborhood U_i of x_i in X, $0 \leq i \leq k-1$, such that $\operatorname{diam}[\operatorname{Cl} U_i] < \delta$ for $0 \leq i \leq k-1$, and $\operatorname{Cl} U_i \cap \operatorname{Cl} U_j = \emptyset$ for $0 \leq i < j \leq k-1$. For each $i \in \{0, \ldots, k-1\}$, we define the map $\varphi_i : \operatorname{Bd} U_i \cup \{x_i\} \to X$ by $\varphi_i = f$ on $\operatorname{Bd} U_i$ and $\varphi_i(x_i) = x_{i+1}$. Since $\operatorname{diam}[\operatorname{Im}\varphi_i] < \xi$ by (2), we have an extension $\tilde{\varphi}_i : \operatorname{Cl} U_i \to X$ of φ_i with $\operatorname{diam}[\operatorname{Im}\tilde{\varphi}_i] < \varepsilon/2$ by (1).

Now we define the map $g : X \to X$ by $g = f$ on $X \setminus \cup_{i=0}^{k-1} U_i$ and $g = \tilde{\varphi}_i$ on $\operatorname{Cl} U_i$ for $0 \leq i < j \leq k-1$. Then it is easy to see that $d(f, g) < \varepsilon$ and $x \in \operatorname{Per}(g)$. □

The second is the result by Block and Franke [4, Theorem H], which characterizes the case where all chain recurrent points are nonwandering, in terms of stability of the nonwandering set under perturbations:

Theorem ([4] for manifolds). *Let (X, d) be an n-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connectedness assumption), and $f : X \to X$ be a map. Then $\Omega(f) = \mathrm{CR}(f)$ if and only if f does not permit Ω-explosions; that is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that if $g : X \to X$ with $d(f, g) < \delta$, then each point of $\Omega(g)$ belongs to the ε-neighborhood of $\Omega(f)$, where $\Omega(h)$ means the nonwandering set of a map h.*

It is hence quite important to know how large the set $\mathrm{CR}(f)$ is. In many systems the chain recurrent set indeed turns out to be small, for example, Franzová [9] proved that if X denotes the interval then for a generic (in the uniform metric) continuous maps the chain recurrent set has Lebesgue measure zero.

2. 鎖回帰集合の測度零性

We now give the terminology and notation needed in what follows. A *map on X* is a continuous function $f : X \to X$ from a space X to itself; f^0 is the identity map, and for every $n \geq 0$, $f^{n+1} = f^n \circ f$. The dimension $\dim X$ of a space X means the covering dimension (see [8] and [12]). By a *graph*, we mean a *connected one-dimensional* compact polyhedron. We let $f : X \to X$ be a map from a compact metric space (X, d) to itself. Let $x, y \in X$. An *ε-chain* from x to y is a finite sequence of points $\{x_0, x_1, \ldots, x_n\}$ of X such that $x_0 = x$, $x_n = y$ and

$d(f(x_{i-1}), x_i) < \varepsilon$ for $i = 1, \dots, n$. We say x *can be chained to* y if for every $\varepsilon > 0$ there exists an ε-chain from x to y, and we say x is *chain recurrent* if it can be chained to itself. The set of all chain recurrent points is called the *chain recurrent set* of f and denoted by $\mathrm{CR}(f)$. The chain recurrent set is non-empty, closed in X and f-strongly invariant, and the set depends only on the topology. A point $x \in X$ is said to be *wandering* if for some neighborhood V of x, $f^n(V) \cap V = \emptyset$ for all $n > 0$. The set of points which are not wandering is called the *nonwandering set* and denoted by $\Omega(f)$.

We state fundamental facts from geometric topology. A space X is said to be *locally* $(n-1)$*-connected* if for every $x \in X$ and every neighborhood U of x in X, there exists a neighborhood $V \subseteq U$ of x in X such that every map $f : S^k \to V$ extends to a map $\tilde{f} : B^{k+1} \to U$ for every $0 \leq k \leq n-1$, where S^k and B^{k+1} stand for the unit k-dimensional sphere and the unit $(k+1)$-dimensional ball of the $(k+1)$-dimensional Euclidean space, respectively.

Here is our main result.

Theorem 2.1 ([15]). *Let (X, d) be an n-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$ we simply skip the local connectedness assumption), and μ be a finite Borel measure on X without atoms at the isolated points of X. Then the set of maps on X with the chain recurrent set of μ-measure zero is residual in the space of all maps on X.*

Remark 1.
(1) The interval case modulo Lebesgue measure of the theorem above was proved by Franzová [9].
(2) Analogous results to Theorem 2.1, Corollary 2.2 and Theorem 3.1 (below) hold for the nonwandering set of a map.
(3) The main theorem is false if μ has an atom at the isolated points of X.
(4) It is well known that any f-invariant finite measure μ is supported by the set of recurrent points ([14]). In particular $\mu(\mathrm{CR}(f)) > 0$. This implies that with all the assumptions of Theorem 2.1, a generic map f does not preserve a given finite measure μ.

We note that a manifold and a polyhedron are locally contractible. The *n-dimensional universal Menger compactum* M_n^{2n+1} is obtained by a process of successively deleting cubes from the $(2n+1)$-cube (see [8, p. 96], [2], [11]). When $n = 0$, we obtain the Cantor set, and when $n = 1$, the Menger curve (which is referred to as the Menger sponge in the fractal literature). A compact *n-dimensional Menger manifold* is a compact metric space locally homeomorphic to the n-dimensional universal Menger compactum M_n^{2n+1}. A topological characterization of a compact n-dimensional Menger manifold obtained by Bestvina [2] (cf. Anderson [1] for $n = 1$) is: a compact metric space X is an n-dimensional Menger manifold if and only if it is n-dimensional, locally $(n-1)$-connected, and satisfies the disjoint n-cells property. Kato, Kawamura, Tuncali and Tymchatyn [11] studied measure theoretic properties of the dynamics of Menger manifolds.

Corollary 2.2 ([15]). *Let X be a compact and n-dimensional either manifold, Menger manifold or polyhedron with no isolated points, where $n \in \mathbb{N}$, and μ be a finite Borel measure on X. Then the set of maps on X with the chain recurrent set of μ-measure zero is residual in the space of all maps on X.*

3. 鎖回帰集合の連結性

We give an application of the main theorem to dynamical systems of graph maps.

Theorem 3.1 ([15]). *Let G be a graph. Then the set of maps on G with the chain recurrent set being totally disconnected is residual in the space of all maps on G.*

Motivated by the result above, we discuss the relation between the chain recurrent set and its connectivity. We need some definitions. A map $f : X \to X$ is said to be *chain transitive* if for every $x, y \in X$, x can be chained to y.

The next is a slight extension of Theorem 2.8 in [6] to the case of the chain recurrent sets of arbitrary surjective maps.

Proposition 3.2 ([15]). *Let $f : X \to X$ be a surjective map on a compact metric space (X, d). If the restriction $f|_{\mathrm{CR}(f)} : \mathrm{CR}(f) \to \mathrm{CR}(f)$ is chain transitive, then $\mathrm{CR}(f) = X$.*

Proposition 3.3 ([15]). *Let $f : X \to X$ be a surjective map on a compact metric space (X, d). If the chain recurrent set $\mathrm{CR}(f)$ of f is connected, then $\mathrm{CR}(f) = X$.*

Remark 2. If $f : X \to X$ is *surjective* and $\mathrm{CR}(f) \neq X$, then $\mathrm{CR}(f)$ must be disconnected by Proposition 3.3. Using a similar argument to that in the proof (without measurable argument) of Theorem 2.1, the property $\mathrm{CR}(f) \neq X$ is generic if X is an n-dimensional locally $(n-1)$-connected compact metric space, where $n \geq 0$ (for $n = 0$, skip the local connected condition, but on further condition "with an accumulation point").

以上のことにより, 連結性に関する次の問いは自然であるが, この話題についてはまた別の機会としたい。

Question. Is a totally disconnected property of the chain recurrent set generic?

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